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# Unified approach to crossover phenomena

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A general analytical method is developed for describing crossover phenomena of arbitrary nature. The method is based on the algebraic self-similar renormalization of asymptotic series, with control functions defined by crossover conditions. The method can be employed for such difficult problems for which only a few terms of asymptotic expansions are available and no other techniques are applicable. As an illustration, analytical solutions for several important physical problems are presented. [S1063-651X(98)14009-6]

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## I. INTRODUCTION

Crossover phenomena are ubiquitous in nature. Probably, they are much more common than phase transitions. When speaking about crossover phenomena, one usually keeps in mind the following picture: A function $f(x)$, describing a physical quantity, is continuous in an interval $x_1 \leqslant x \leqslant x_2$, but the behavior of $f(x)$ in the vicinity of the boundaries of this interval is *qualitatively different* near $x_1$ as compared to $x_2$. The qualitative change of the behavior of this function, as $x$ moves from one side to another side, is commonly understood as a crossover.

It is possible to quote hundreds of examples of different crossovers. For instance, many physical quantities qualitatively change their behavior when passing from the weak-coupling to the strong-coupling limit [1]. This concerns, e.g., the majority of problems having to do with the behavior of energies as functions of a coupling parameter in statistical physics, quantum mechanics, and field theory. Let us mention in this respect the dependence of the spectra of Schrödinger operators on the anharmonicity parameter for variegated anharmonic models. The energy spectrum of such models is qualitatively different in the weak-coupling (weak anharmonicity) as compared to the strong-coupling (strong anharmonicity) limits.

A famous example of a crossover phenomenon is the Kondo effect [2] when the behavior of a system changes qualitatively at varying temperature. Although this transformation goes smoothly, with no discontinuities in thermodynamic characteristics, the change of properties is so noticeable that one can ascribe a particular point, called the Kondo temperature, to a region dividing qualitatively different regimes of low and high temperatures.

Another renowned example of a crossover is the Fröhlich polaron problem [3]. Polaron characteristics, such as its energy or effective mass, change qualitatively when varying the coupling parameter describing electron-phonon interactions. This change happens so explicitly that for about two decades there were many speculations suggesting that there exists a phase transition at a particular value of the coupling parameter. However, modern highly accurate Monte Carlo calculations [4] confirm the initial Feynman picture [3], proving that we meet here not a phase transition but a classical crossover.

In the examples mentioned above, of simple anharmonic models, the Kondo effect, and of the Fröhlich polaron problem, the crossover, when varying a coupling parameter or temperature, is monotonic. However, there are cases when crossover is not monotonic. This concerns, for instance, one-dimensional antiferromagnet whose characteristics are considered as functions of spin. Then the Haldane gap [5], as spin changes from small to large values, exhibits a very nonmonotonic behavior becoming zero at each half–odd-integer spin.

A nonmonotonic behavior can often be met in the dependence of spectra of collective excitations on a wave vector. Then the crossover from the region of small wave numbers, corresponding to the long-wavelength acoustic regime, to the region of large wave vectors, corresponding to a single-particle regime, can go through a nontrivial intermediate region displaying maxima and minima, associated with maxons and rotons [6–8].

We could adduce a number of other examples of crossover phenomena related to interesting physical problems. Let us just mention deconfinement in nuclear matter, which is rather a crossover phenomenon than a phase transition (see the discussion in [9]). However, we think that it is already clear that crossover phenomena are widespread in nature and that it is important to know how to describe them.

The description of crossover phenomena occurring in realistic statistical systems is usually very complicated. This is because one needs to find physical characteristics for a wide range of parameters, which is far from being trivial for complex systems. For example, we have to find a function $f(x)$ on the whole semiaxis $0 \leqslant x < \infty$. The variable $x$ may represent, e.g., a coupling parameter, temperature, or wave vector. Quite often, one can define, more or less easily, the asymptotic behavior of $f(x)$ near the boundaries of the interval $[0,\infty)$, that is, when $x \to 0$ and $x \to \infty$. Such asymptotic expressions may correspond to the weak-coupling and

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strong-coupling limits, to the low-temperature and high-temperature expansions, to the long-wavelength and short-wavelength approximations, and so on. However, in the intermediate region, where there are no small parameters, one cannot invoke perturbative techniques. It would be nice to possess a method allowing one to construct interpolation formulas only from the knowledge of asymptotic expansions near boundaries.

There exist several summation techniques, such as Padé approximation, Borel summation, and conformal mapping, that permit one to ascribe effective sums to asymptotic series [1,10]. However, all these techniques are not applicable in principle for the complex problems we are interested in here. This is because of the following main reasons. First, all these summation techniques, to be applicable, require the knowledge of tens of terms in an asymptotic series. Such a luxurious information is usually not available for nontrivial systems, for which standardly one is able to derive just a few perturbative terms. Then all mentioned summation techniques are useless. Second, the latter are just *summation* methods, while we here are concerned with an *interpolation* problem. Summation and interpolation are far from being the same.

The most known interpolation method is the *two-point* Padé approximation [11], which should not be confused with the standard Padé approximation [10]. However, the former method, being a derivative of the latter, shares all its deficiences. Among the most important shortcomings of Padé approximants we mention the following: the necessity of having many perturbative terms, the appearance of unphysical poles, the ability to deal only with the so-called compatible variables, the possibility of describing only those functions that have at infinity a power-law behavior with rational powers, and the impossibility to correctly treat nonmonotonic crossover. These difficulties are well known and repeatedly discussed in literature [10–18]. In addition, we remind the reader that the Padé approximation is rather a numerical method.

In the present paper we advance an *analytical* approach for treating interpolation problems of arbitrary nature. This approach is free of the shortcomings typical of the two-point Padé approximation. What also makes this approach more general than any other known methods is the possibility of using it for those difficult cases when just a few asymptotic terms are available and no other method is applicable. We illustrate the approach by applying it to several difficult problems with monotonic as well as nonmonotonic crossover. We would like to stress that the physical problems we consider not only illustrate the wide applicability of the suggested approach but are also of interest as such. Therefore, the interpolation formulas we derive present analytical solutions for important physical problems.

## II. GENERAL APPROACH

The interpolation approach we advance here is based on the ideas of our previous papers. However, since we do not assume that a reader in the common audience is already well acquainted with these ideas, we provide here a clearly understandable description of the method in general, complementing it by those particulars that are necessary for adjusting it to the interpolation problem.

Assume that we are looking for a physical characteristic presented by a function $f(x)$, in which the variable $x$ changes in the interval $[0,\infty)$. The standard situation is when the physical problem under investigation is so complicated that it is difficult, or even impossible, to find a reasonable approximation for the sought function in the whole given interval. However, it is often feasible to get an asymptotic expansion for small variables

$$f(x) \simeq p_k(x) \quad (x \to 0), \tag{1}$$

where $k=0,1,2,\ldots,$ employing a kind of perturbation theory. Also, it is often possible to find an asymptotic behavior of the function at large variables, say,

$$f(x) \simeq f_{as}(x) \quad (x \to \infty). \tag{2}$$

Then the interpolation problem consists in answering the following question: What can be said about the behavior of the function in the whole interval $[0,\infty)$ being based on the asymptotic information in Eqs. (1) and (2)? Usually, not much, since the asymptotic expressions (1) and (2), being derived in two opposite limits, have nothing in common with each other. In addition, perturbative approximations, such as Eq. (1), usually result in divergent series. When one is lucky enough, dealing with a more or less easy case, so that tens of perturbative terms in Eq. (1) could be calculated, one could invoke some known summation technique in order to ascribe an effective sum to a divergent series. However, even in such a lucky case, the found effective sum may have, and usually has nothing to do with, the limit (2) from another side of the axis $[0,\infty)$. For example, the standard case is when the limit (2) corresponds to an exponential behavior. If one uses Padé approximants or any other techniques based on them, for instance, Padé-Borel summation [1], one comes to an effective sum in the form of rational fractions, which cannot be matched with an exponential. In the less lucky but more realistic case, when only a few perturbative terms are known, all these summation techniques become in principle useless. How could we proceed in such difficult cases in order to find an interpolation formula connecting Eqs. (1) and (2)?

The first thing we need to do is to understand how to extract useful information from a divergent sequence $\{p_k(x)\}$ when only a few initial terms of it are available. It would be nice to reconstruct the sequence $\{p_k(x)\}$ in such a way as to improve its convergence properties. Having only a few terms, we cannot resort to the standard summation techniques. Nevertheless, a reconstruction is possible with the help of *control functions* [19,20]. Let us denote the procedure of introducing control functions as

$$C_s\{p_k(x)\} = P_k(x,s), \tag{3}$$

where $s = s_k(x)$ is a set of functions such that the sequence $\{P_k(x,s_k)\}$ has better convergence properties than $\{p_k(x)\}$. The name ''control functions'' reflects their role in controlling convergence. The introduction of such functions can be done in several ways. Generally, any procedure of obtaining a sequence of approximations consists of three elements: a calculational algorithm, an initial approximation, and additional transformations. For example, by introducing a relaxation or damping parameter into the numerical Newton method, one can improve the convergence of the latter [21]. Under a given calculational algorithm, one may include control functions in an initial approximation $P_0(x,s)$, after which all the following approximations also become dependent on $s$. This variant of introducing control functions is probably the most widely used. One takes an initial Hamiltonian, Lagrangian, or action as depending on trial parameters that are defined as control functions by imposing an addition condition such as the minimal-difference condition [19,20,22–26] or the minimal-sensitivity condition [27–32]. Such conditions are, of course, heuristic. For simple cases, such as zero-dimensional and one-dimensional oscillators, for which perturbative terms of arbitrary order can be obtained, one may define control functions directly from the condition of convergence of these terms, as $k\to\infty$ [33–35]. Finally, if a calculational algorithm with an initial approximation have been fixed, one may introduce control functions by subjecting the resulting asymptotic series to additional transformations. These can be either a change of variables, with the reexpansion of the given series in powers of new variables, or a transformation of a series itself. An example of the former case is the order-dependent mapping and of the latter is the Borel-Leroy transformation [1]. However, these transformations require the knowledge of the analyticity properties of the sought functions itself, which is rarely available.

To our mind, a transformation that one wishes to apply to an asymptotic series in order to construct an analytical approach must satisfy three main stipulations: (i) It must be *general* to be applicable to any function without requiring the knowledge of its properties that are not known. The sole assumption involved should be the existence of the sought function. (ii) It must be *simple* to permit an analytical investigation. At the same time, simplicity is usually a requisite for generality. (iii) It must be *invertible*, with a uniquely defined inverse transformation. This is evidently necessary to return from a transform to the function itself. In addition, it would be desirable to have an apparent interpretation of the meaning of the chosen transformation.

These stipulations are satisfied by the algebraic transformation [36–38] whose general form is

$$C_s\{f(x)\}=a(x,s)+b(x,s)f(x), \tag{4}$$

where $a(x,s)$ and $b(x,s)$ are any functions guaranteeing the uniqueness of the inverse transformation

$$C_s^{-1}[C_s\{f(x)\}]=\frac{C_s\{f(x)\}-a(x,s)}{b(x,s)}=f(x).$$

One of the simplest variants of Eq. (4), as applied to a term $p_k(x)$ of a sequence $\{p_k(x)\}$, is

$$C_s\{p_k(x)\}=P_k(x,s)=x^s p_k(x). \tag{5}$$

This variant is not only simple but also has a transparent meaning when $p_k(x)$ is a $k$th-order truncated series in powers of $x$. Then the transformation (5) effectively increases the approximation order from $k$ to $k+s$.

Assume that, in some way, we have introduced control functions constructing from an initial sequence $\{p_k(x)\}$ a transformed sequence $\{P_k(x,s)\}$ with better convergence properties. Now we have to concretize in what sense the properties of $\{P_k(x,s)\}$ should be better than those of $\{p_k(x)\}$. The greatest achievement would be if the transformed sequence $\{P_k\}$ is such that we could notice a relation between subsequent terms $P_k$ and $P_{k+1}$. If so, we would be able to map the low-order terms to those of arbitrary high order. That is, having just a few initial terms of a sequence $\{P_k\}$, we could extrapolate them to higher orders of $k$ defining an effective limit $P^*$ of this sequence. To formulate a relation between subsequent terms of a sequence of approximations means to define the property of self-similarity between these terms. This can be called the *approximation self-similarity* [39–43]. To formulate the latter, we need to invoke some further transformations. To this end, let us define an expansion function $x(\varphi,s)$ by the equation

$$P_0(x,s)=\varphi, \quad x=x(\varphi,s). \tag{6}$$

Then we introduce an endomorphism

$$y_k(\varphi,s)\equiv P_k(x(\varphi,s),s), \tag{7}$$

with an initial condition

$$y_0(\varphi,s)\equiv\varphi, \tag{8}$$

following from Eq. (6). The transformation inverse to Eq. (7) reads

$$P_k(x,s)=y_k(P_0(x,s),s). \tag{9}$$

By these definitions, the sequence $\{y_k(\varphi,s)\}$ is bijective to $\{P_k(x,s)\}$. The property of self-similarity between the terms of the sequence $\{y_k(\varphi,s)\}$ is written as [39–43]

$$y_{k+p}(\varphi,s)=y_k(y_p(\varphi,s),s). \tag{10}$$

This is nothing but the semigroup property $y_{k+p}=y_k y_p$. The relation (10) is reminiscent of a functional equation of the renormalization group [1,44]. However, here there is a principal difference. Renormalization-group equations [1,44] relate a function with scaled variables with the function itself. So a renormalization-group equation describes motion with respect to function variables. In our case, Eq. (10) relates different approximations from the sequence $\{y_k\}$. Therefore, the self-similarity of the approximation (10) defines motion with respect to approximation orders that play the role of discrete time. In the language of dynamical theory, a dynamical system with discrete time is called a cascade. Since the trajectory $\{y_k(\varphi,s)\}$ of this cascade is, by construction, bijective to the sequence of approximations $\{P_k(x,s)\}$, a family of endomorphisms $\{y_k|k=0,1,2,\dots\}$ can be named the *approximation cascade* [45,46]. An important feature of this cascade is that the self-similarity of the approximation (10) is a necessary condition for the fastest convergence [42,43].

For the purpose of developing an analytical theory, it is not convenient to deal with discrete time. It would be desirable to pass from the discrete index $k=0,1,2,\dots$ to a continuous variable $t\in[0,\infty)$. This can be done [39–42] by introducing an endomorphism $y_t(\varphi,s)$ such that $y_t$ has the same group property

$$y_{t+\tau}(\varphi,s) = y_t(y_\tau(\varphi,s),s) \tag{11}$$

as $y_k$ in Eq. (10) and the values

$$y_t(\varphi,s) = y_k(\varphi,s) \quad (t=k) \tag{12}$$

at integer $t$ coincide. The so-defined family of endomorphisms $\{y_t | t \in [0,\infty)\}$ forms an *approximation flow* and the conditions (11) and (12) define the *embedding* of a cascade into flow [45,46]. From the relation (11) with continuous time, it is easy to derive the Lie evolution equation

$$\frac{\partial}{\partial t} y_t(\varphi,s) = v_t(y_t,s), \tag{13}$$

with the velocity field

$$v_t(y_t,s) = \lim_{\varphi \to y_t} \lim_{t \to 0} \frac{\partial}{\partial t} y_t(\varphi,s). \tag{14}$$

Equation (13) can be rewritten in the integral form

$$\int_{y_t}^{y_{t+\tau}} \frac{d\varphi}{v_t(\varphi,s)} = \tau. \tag{15}$$

To study the properties of an approximation flow, we may invoke powerful techniques of dynamical theory [47–50].

What we need to obtain at the end is an effective limit of the sequence $\{P_k(x,s)\}$. Since the latter is bijective to the trajectory $\{y_k(\varphi,s)\}$ of the approximation cascade, the limit of $\{P_k\}$ is in a one-to-one correspondence with a stable fixed point of the cascade [45,46]. A fixed point is defined as a zero of velocity. The cascade velocity can be written as the Euler discretization of the flow velocity [42,43], which reads

$$v_k(\varphi,s) = y_{k+1}(\varphi,s) - y_k(\varphi,s) + \Delta s \frac{\partial}{\partial s} y_k(\varphi,s), \tag{16}$$

where $\Delta s$ is a variation of a control function. Since this variation is not known, we cannot find an exact zero of the velocity (16), but can find only its approximate zero defining a quasifixed point. For instance, we may set

$$\Delta s \frac{\partial}{\partial s} y_k(\varphi,s) = 0, \tag{17}$$

which is satisfied if either $\Delta s = 0$ or $\partial y_k / \partial s = 0$. In both the cases, the velocity (16) becomes

$$v_k(\varphi,s) = y_{k+1}(\varphi,s) - y_k(\varphi,s). \tag{18}$$

This and several other ways of defining quasifixed points and the related velocities have been analyzed in detail in Refs. [51–53]. The motion in the space of approximations, near a quasifixed point, is described by the evolution integral (15), which can be written as

$$\int_{P_k}^{P_{k+1}^*} \frac{d\varphi}{v_k(\varphi,s)} = \tau, \tag{19}$$

where $P_k = P_k(x,s)$, $P_k^* = P_k^*(x,s,\tau)$ is a quasifixed point, and $\tau$ is the minimal time necessary to reach this quasifixed point.

Substituting the cascade velocity (18) into the evolution integral (19), we can find a quasifixed point $P_k^*$. Then we need to make a transformation inverse to the algebraic transformation (4),

$$p_k^*(x,s,\tau) = C_s^{-1}\{P_k^*(x,s,\tau)\}. \tag{20}$$

The resulting approximant (20) is, as is clear from Eq. (19), a function of $P_{k-1}$, that is, of $p_{k-1}$, which can be written as

$$p_k^* = F_k(p_{k-1}).$$

We may repeat the renormalization procedure for $p_{k-1}$, obtaining

$$p_k^* = F_k(F_{k-1}(p_{k-2})),$$

and so on. After $k$ steps of such a procedure, called self-similar bootstrap [38], we arrive at

$$p_k^* = F_k(F_{k-1}(\cdots F_1(p_0))\cdots).$$

In short notation, the latter can be presented as a quasifixed-point equation

$$p_k^* = F_k(p_{k-1}^*). \tag{21}$$

As long as Eq. (21) implies a $k$-step renormalization, the resulting $p_k^*$ will contain two sets

$$\overline{s_k} = \{s_1, s_2, \ldots, s_k\}, \quad \overline{\tau_k} = \{\tau_1, \tau_2, \ldots, \tau_k\}$$

of $2k$ control functions, which can be denoted as

$$p_k^* = p_k^*(x, \overline{s_k}, \overline{\tau_k}).$$

Now it is time to recall the main aim of the present paper, that is, to suggest an approach for treating crossover phenomena. Therefore, we must remember the asymptotic condition (2) and require that the found approximation (21) satisfy the condition

$$p_k^*(x, \overline{s_k}, \overline{\tau_k}) \simeq f_{as}(x) \quad (x \to \infty). \tag{22}$$

This defines the sets $\overline{s_k} = \overline{s_k}(x)$ and $\overline{\tau_k} = \overline{\tau_k}(x)$ of control functions. With the found control functions, we obtain the final self-similar approximant

$$f_k^*(x) = p_k^*(x, \overline{s_k}(x), \overline{\tau_k}(x)). \tag{23}$$

What makes the present paper different from our previous publications is the *systematic use of the asymptotic conditions* of type (22) for defining control functions. The suggested procedure is designed to self-similarly connect the left and right asymptotic expansions of a function on a given interval. For concreteness, we have considered above the connectinig procedure from the left to the right. However, as is evident, the same procedure can be followed from the right to the left, that is, starting from an asymptotic expansion at the right boundary of the interval $[0,\infty)$, when $x \to \infty$, and then connecting the obtained approximant with the asymptotic form at the left boundary, where $x \to 0$. In any case, we shall arrive at an approximant whose structure is governed by the quasifixed-point equation (21).

To show explicitly what the structure of the approximant $p_k^*$ is, let us take an initial expansion $p_k(x)$, as $x \to 0$, in the form of a standard power series

$$p_k(x) = \sum_{n=0}^{k} a_n x^n. \tag{24}$$

Employing the algebraic transform (5), we have

$$P_k(x,s) = \sum_{n=0}^{k} a_n x^{n+s}. \tag{25}$$

Equation (6) reads

$$P_o(x,s) = a_0 x^s = \varphi, \tag{26}$$

from which the expansion function is

$$x = \left(\frac{\varphi}{a_0}\right)^{1/s}. \tag{27}$$

For the endomorphism (7) we get

$$y_k(\varphi,s) = \sum_{n=0}^{k} a_n \left(\frac{\varphi}{a_0}\right)^{1+n/s}. \tag{28}$$

The cascade velocity (18) becomes

$$v_k(\varphi,s) = a_{k+1}\left(\frac{\varphi}{a_0}\right)^{1+(k+1)/s}. \tag{29}$$

Substituting Eq. (29) into the evolution integral (19), we find a quasifixed point $P_k^*$, after which we need to make the inverse transformation (20),

$$p_k^*(x,s,\tau) = x^{-s} P_k^*(x,s,\tau). \tag{30}$$

This results in the expression

$$p_k^*(x,s,\tau) = \left[ p_{k-1}^{-k/s}(x) - \frac{k a_k \tau}{s a_0^{1+k/s}} x^k \right]^{-s/k}. \tag{31}$$

Let us note that when $s \to \infty$,

$$\lim_{s\to\infty} p_k^*(x,s,\tau) = p_{k-1}(x) \exp\left(\frac{a_k}{a_0} \tau x^k\right), \tag{32}$$

which explains how naturally exponentials appear in our approach [38].

The quasifixed-point equation (21), as applied to Eq. (31), gives

$$p_k^* = [(p_{k-1}^*)^{1/n_k} + B_k x^k]^{n_k}, \tag{33}$$

where, for brevity, the arguments of $p_k^* = p_k^*(x,s_k,\tau_k)$ are not written down and the notation

$$n_k \equiv -\frac{s_k}{k}, \quad B_k \equiv \frac{a_k \tau_k}{n_k a_0^{1-1/n_k}} \tag{34}$$

is used. In the same way we get

$$p_{k-1}^* = [(p_{k-2}^*)^{1/n_{k-1}} + B_{k-1} x^{k-1}]^{n_{k-1}}$$

and so on down to

$$p_2^* = [(p_1^*)^{1/n_2} + B_2 x^2]^{n_2}$$

and

$$p_1^* = (p_0^{1/n_1} + B_1 x)^{n_1},$$

where $p_0 = a_0$. This shows that the structure of Eq. (33) is a sequence of nested roots. For instance, a third-order approximant looks like

$$p_3^* = \{[(p_0^{1/n_1} + B_1 x)^{n_1/n_2} + B_2 x^2]^{n_2/n_3} + B_3 x^3\}^{n_3}.$$

Control functions $s_k$ and $\tau_k$ are to be found from the asymptotic condition (22). Because of the relation (34), this is the same as defining the powers $n_k$ and amplitudes $B_k$. The equalities (34) are nothing but a change of variables, so that instead of $s_k$ and $\tau_k$ we may consider $n_k$ and $B_k$ as new control functions. For practical purposes, we may at once write down a $k$th-order approximant in the form of Eq. (33) and directly define $n_k$ and $B_k$ from the condition (22). If the latter gives several solutions for control functions, then we should opt for the solution that leads to the decrease of $B_k$ with increasing $k$. This follows from Eq. (33), from which it is evident that $p_k^*$ tends to a fixed point $p^*$ if and only if $B_k \to 0$ as $k \to \infty$.

Before considering complex physical problems using the method we have described, let us illustrate it by a simple example of the one-dimensional quartic anharmonic oscillator. Consider the dimensionless ground-state energy $e(g)$ as a function of the coupling, or anharmonicity, parameter $g \in [0,\infty)$. In the weak-coupling limit, when $g \to 0$, perturbation theory results [54] in the expansion

$$e(g) \simeq a_0 + a_1 g + a_2 g^2 + a_3 g^3 + a_4 g^4, \tag{35}$$

where

$$a_0 = \frac{1}{2}, \quad a_1 = \frac{3}{4}, \quad a_2 = -\frac{21}{8},$$

$$a_3 = \frac{333}{16}, \quad a_4 = -\frac{30\,885}{128}.$$

In the strong-coupling limit, when $g \to \infty$, the asymptotic behavior is known [55,56] to be

$$e(g) \simeq A_0 g^{1/3} + A_1 g^{-1/3} + A_2 g^{-1} + A_3 g^{-5/3}, \tag{36}$$

with the coefficients

$$A_0 = 0.667\,986, \quad A_1 = 0.143\,669,$$

$$A_2 = -0.008\,628, \quad A_3 = 0.000\,818.$$

Starting from the linear approximation $p_1(g)=a_0+a_1(g)$ from Eq. (35), we find

$$p_1^*(g,n_1,B_1)=(a_0^{1/n_1}+B_1 g)^{n_1},$$

which is the first approximation from Eq. (33). Requiring the validity of the asymptotic condition

$$p_1^*(g,n_1,B_1)\simeq A_0 g^{1/3} \quad (g\to\infty),$$

we find $n_1=\frac{1}{3}$ and $B_1=A_0^3=0.298\,059$. From here we could recalculate $s_k$ and $\tau_k$ using the relations (34); however, this is not necessary since, as is explained above, now $n_k$ and $B_k$ play the role of control functions and what we need finally are exactly $n_k$ and $B_k$. The quantity $n_k$ can be called a *crossover index* and $B_k$ a *crossover amplitude*. With the found $n_1$ and $B_1$, we define, analogously to Eq. (23), the first-order self-similar approximant

$$e_1^*(g)\equiv p_1^*\left(g,\frac{1}{3},A_0^3\right)$$

for the ground-state energy, which is written

$$e_1^*(g)=(a_0^3+A_0^3 g)^{1/3}. \tag{37}$$

Comparing the values of Eq. (37) with numerical results [55] that can be treated as exact, we see that the maximal error, for $g\geqslant 0$, of Eq. (37) is $-6.8\%$, occurring at $g\approx 0.7$.

Similarly, we find the second-order self-similar crossover approximant

$$e_2^*(g)=[(a_0^{9/2}+Cg)^{4/3}+B_2 g^2]^{1/6}, \tag{38}$$

where the two first terms in the asymptotic expansion (36) are used and $C=0.1971$ and $B_2=A_0^6=0.0888$. The maximal error of Eq. (38) is $-2.9\%$ at $g\approx 0.3$. Continuing the procedure, we get the third-order approximant

$$e_3^*(g)=\{[(a_0^{81/14}+C_1 g)^{4/3}+C_2 g^2]^{7/6}+B_3 g^3\}^{1/9}, \tag{39}$$

with $C_1=0.1116$, $C_2=0.0784$, and $B_3=A_0^9=0.026\,48$. The maximal error of Eq. (39) is $-1.7\%$ at $g\approx 0.1$. In fourth order, we obtain

$$e_4^*(g)=(\{[(a_0^{243/35}+D_1 g)^{4/3}+D_2 g^2]^{7/6}+D_3 g^3\}^{10/9} + B_4 g^4)^{1/12}, \tag{40}$$

where $D_1=0.0625$, $D_2=0.050\,05$, $D_3=0.0131$, and $B_4=A_0^{12}=0.007\,89$. The maximal error of Eq. (40) is $-1.3\%$ at $g\approx 0.1$. The sequence is uniformly convergent, which can be seen from the monotonic decrease of errors from about 7% to 1%.

We would like to emphasize that our aim in the present paper is to suggest a *systematic analytical* method permitting one to derive explicit expressions describing crossover phenomena. The advantage of having accurate analytical formulas, as compared to numerical results of numerical methods, is in the simplicity of analyzing such formulas with respect to the variation of parameters entering these formulas. Also, having an analytical formula corresponding to a measurable quantity often gives more information about the studied system than just numbers. As an example, we may mention the geometric spectral inversion in quantum mechanics [57,58].

## III. FRÖHLICH POLARON

The Fröhlich optical polaron problem [3] is an interesting physical example of a crossover about which there existed a controversy lasting for around 30 years. Some researchers, analyzing the polaron ground-state energy $e(\alpha)$ as a function of the electron-phonon coupling parameter $\alpha$, found an indication of a phase transition from a state of a freely moving weak-coupling polaron to a localized state of a strong-coupling polaron (see the discussion in [15]). One such indication has been suggested by Gross [59]. However, as modern investigations show [4], there is no phase transition in the polaron problem, but the latter is an example of a classical crossover between the weak-coupling and strong-coupling limits.

In the weak-coupling limit, the ground state of the polaron has an asymptotic behavior

$$e(\alpha)\simeq a_1\alpha+a_2\alpha^2+a_3\alpha^3 \quad (\alpha\to 0), \tag{41}$$

with three well-established terms [15,60,61], in which

$$a_1=-1, \quad a_2=-1.591\,962\times 10^{-2},$$

$$a_3=-0.806\,070\times 10^{-3}.$$

In the strong–coupling limit, Miyake [62,63] obtained

$$e(\alpha)\simeq A_0\alpha^2+A_2+A_4\alpha^{-2} \quad (\alpha\to\infty), \tag{42}$$

where

$$A_0=-0.108\,513, \quad A_2=-2.836, \quad A_4=-4.864.$$

The terms of the weak-coupling expansion are known here with a better precision than those of the strong–coupling expansion. In addition, the coefficient $a_k$ decreases as $k$ increases, while $A_k$ increases with $k$. Therefore, here we have to construct self-similar approximations from the right to the left, that is, starting from the perturbative expression (42), we find a self-similar approximant $e_k^*(\alpha)$, with control functions defined from the asymptotic condition

$$e_k^*(\alpha)\simeq e_{as}(\alpha) \quad (\alpha\to 0),$$

in which $e_{as}(\alpha)$ is given by Eq. (41). The accuracy of the found self-similar approximants $e_k^*(\alpha)$ can be evaluated by comparing them with the values $e(\alpha)$ obtained by Monte Carlo numerical calculations [4,64]. As usual, the accuracy of self-similar crossover approximants is the worst in an intermediate region, where a weak-coupling expansion is connected with a strong-coupling one. For the polaron energy $e_k^*(\alpha)$, the maximal error occurs at $\alpha\approx 10$.

The first-order crossover approximation gives

$$e_1^*(\alpha)=-\alpha(1+B\alpha^2)^{1/2}, \tag{43}$$

with $B=A_0^2=0.011\,775$. The maximal error of Eq. (43) is $-10.5\%$. The second-order approximant is

$$e_2^*(\alpha) = -\alpha[1+\alpha(B_0+B_2\alpha^2)^{3/2}]^{1/4}, \tag{44}$$

where $B_0 = 0.159\,468$ and $B_2 = A_0^{8/3} = 0.002\,679$. The maximal error of Eq. (44) is $-4.54\%$. Finally, we may find the third-order self-similar approximant by taking account of all three known terms in Eq. (41). This yields

$$e_3^*(\alpha) = -\alpha\{1+\alpha[C_0+\alpha(C_1+C_2\alpha^2)^{3/2}]^{5/4}\}^{1/6}, \tag{45}$$

where $C_0 = 0.152\,804$, $C_1 = 0.049\,617$, and $C_2 = A_0^{16/5} = 0.000\,819$. The maximal error of Eq. (45) is $-1.5\%$. Again, we see that, with increasing order of the approximation, the accuracy of the found crossover approximants improves from an error of about 10% to about 1%. The very simple formula (45) gives the same accuracy as the Feynman variational calculations [3,65].

## IV. KONDO EFFECT

One of the most remarkable examples of crossover phenomena is given by the Kondo effect [2]. The behavior of the system, consisting of a local impurity spin and conduction electrons, interacting by means of an antiferromagnetic exchange of strength $J$, changes from asymptotically free at high temperatures to that of the impurity screened by an electronic lump, via the crossover region whose onset is characterized by the Kondo temperature estimated as $T_k = D\exp(-1/2J)$, where $D$ stands for the Fermi energy of electrons. We consider below only the case of a single-channel Kondo model with the impurity local moment equal to 1/2. Most of our knowledge about the problem comes from the exact Bethe-ansatz solution [66,67], from the field-theoretic renormalization group (RG) [68–70], and from the Wilson numerical renormalization group [71]. It was pointed out in Ref. [72] that the Bethe-ansatz solution cannot be extrapolated beyond the coupling constant $J$ of order one (see also [73]). Field-theoretic RG results are valid only at $J<1$ as well. On the other hand, the strong-coupling limit $J\to\infty$ of the Kondo model was considered in Ref. [72]. Only the numerical RG treatment of the Kondo problem is valid, formally, for arbitrary $J$. We suggest below a simple analytical approach valid for arbitrary $J$.

Within the framework of the field-theoretic RG in its application to the Kondo crossover, the central role is played by the so-called invariant charge or effective electron-electron coupling $J_{inv}$ [68–70], measuring the intensity of electron-electron interactions via the impurity spin. The field-theoretic Gell-Mann-Low $\beta$ function could be defined using the perturbation theory in the weak-coupling limit [68–70]

$$\beta(J) \simeq -2J^2 + 2J^3 \quad (J \ll 1) \tag{46}$$

or by means of a sophisticated bosonization technique in the strong-coupling limit [72]

$$\beta(J) \simeq -c, \quad c \approx 0.377 \quad (J\to\infty). \tag{47}$$

The left crossover approximation, satisfying by design both known limits, can be obtained giving an improved, self-similarly renormalized Gell-Mann-Low function

$$\beta^*(J) = -2J^2\left(1+\frac{\tau}{2}J\right)^{-2}, \quad \tau \equiv \sqrt{\frac{8}{c}} = 4.607, \tag{48}$$

and $J_{inv}$ is given [68–70] by the equation

$$\int_J^{J_{inv}} \frac{dg}{\beta^*(g)} = \ln\left(\frac{\omega}{D}\right).$$

The last integral can be calculated explicitly and the result may be presented in the form

$$\Phi(J_{inv}) = \ln\left(\frac{\omega}{T_k}\right),$$

$$\Phi(z) = \frac{1}{2z} - \frac{\tau}{2}\ln(z) - \frac{\tau^2}{8}\, z, \tag{49}$$

where $\omega$ stands for the typical external parameters of the problem (temperature and magnetic field) and $T_k$ is the typical internal energy scale, or the Kondo temperature

$$T_k = DJ^{\tau/2}\exp\left(-\frac{1}{2J}+\frac{\tau^2}{8}J\right). \tag{50}$$

This expression for the Kondo temperature has the same form as the famous Wilson numerical RG result

$$T_k = \tilde{D}(J)(2J)^{1/2}\exp\left[-\frac{1}{2J}+1.5824(2J)\right],$$

where $\tilde{D}(J)$ is known to have a power series expansion in $J$ [71]. The origin of the linear correction in the exponential can be traced, therefore, to the strong-coupling limit. To our knowledge, other analytical approaches, including the Bethe-ansatz solution, cannot capture it. The effective interaction determined by Eq. (49) increases to infinity as $\omega$ goes to zero, in agreement with the numerical RG and strong-coupling limit [71,72,74].

## V. ONE-DIMENSIONAL ANTIFERROMAGNET

Extreme caution is needed when any kind of perturbative or nonperturbative approach is applied to the one-dimensional Heisenberg antiferromagnet of arbitrary spin $S$. Even such a general method as the Bethe ansatz fails for $S > 1/2$. Nevertheless, the crossover approach can be of use in this situation.

### A. Autocorrelation function

The Bethe ansatz, despite its failure in the general case of $S>1/2$, allows one to find the magnetic properties of the Heisenberg antiferromagnetic (AF) spin chains of arbitrary spin, when a maximum of two deviations is allowed from the completely aligned (ferromagnetic) state [75]. The magnetization curve and pair correlations had been obtained explicitly for a strong magnetic field, close to the spin-flip transition. The expression for the autocorrelation function $\Theta_0 = \langle S_0^z S_0^z\rangle$, as a function of the number $N$ of spins $S$, has a very simple and transparent form. In Ref. [75], an equivalent quantity

$$F_0(N)=\frac{\Theta_0}{S^2}-\left(\frac{S_T^z}{NS}\right)^2$$

is presented (and compared to numerical data) as a function of the parameter $\sigma$ (demagnetization),

$$\sigma\equiv 1-\frac{S_T^z}{NS}.$$

Here $S_T^z$ stands for the magnetization of a spin-flip phase and is controlled by the magnetic field $h$, so that close to the saturation field $h_s=4S$ [75] one has

$$\frac{S_T^z}{NS}=1-\frac{2}{\pi S}\left(1-\frac{h}{h_s}\right)^{1/2}.$$

Finally, $F_0$ is presented in the form

$$F_0(\sigma)\simeq\frac{\sigma}{S}-\sigma^2+\frac{1}{8}\pi^2S^2(2S-1)^2\sigma^4 \quad (\sigma\ll 1). \tag{51}$$

The last term in Eq. (51), proportional to $(2S-1)^2$, clearly distinguishes an extra contribution from the so-called $C$ states, typical for $S\geqslant 1$ and absent for $S=1/2$, with high probability of having two spin deviations on the same site [75]. An exact, independent of $N$, value

$$F_0=\frac{2}{3} \quad (\sigma=1,\ S=1)$$

is known too [75] and can be used as an asymptotic condition. Let us continue the expansion (51) from the region of small $\sigma$ to the region of $\sigma\sim 1$, along the stable trajectory, ending at $\sigma=1$ at the value $F_0=\frac{2}{3}$. In order to extend the validity of Eq. (51) for $S\geqslant 1$ let us add to Eq. (51) one more trial term $\sim-\sigma^6$ and find the corresponding effective time $\tau$ from the crossover condition at the boundary point. The self-similar bootstrap procedure leads to the crossover approximations

$$F_0^*(\sigma)=\frac{\sigma}{S(1+\sigma S)}, \quad S=\frac{1}{2}, \tag{52}$$

$$F_0^*(\sigma)=\frac{\sigma}{S}\exp\left\{-\sigma S\exp\left[-\frac{1}{8}\pi^2S^2(2S-1)^2\sigma^2\times\exp\left(-\frac{A\sigma^2}{S^2(2S-1)^2}\right)\right]\right\}, \quad S\geqslant 1, \tag{53}$$

where $A=8\tau/\pi^2=0.253$. At $S=1$, $F_0^*(\sigma)$ agrees both qualitatively and quantitatively with the data of Fig. 3 from Ref. [75], the maximal error being $\approx 4\%$. The behavior of $F_0^*(\sigma)$ for $S=1/2$ and $S=1$ is qualitatively similar (universal regime) only as $\sigma\to 0$ $(h\to h_s)$, where $C$ states are suppressed by a magnetic field and is different for all finite $\sigma(h<h_s)$ due to the contribution from $C$ states (nonuniversal regime). The onset of the regime dominated by $C$ states may be related to the inflection point of the curve $F_0^*(\sigma)$ for $\sigma\sim 0.5$, emerging for $S=1$ and absent for $S=1/2$.

Consider the case of $h\to h_s$, $S=1$. Upon a rapid (instant) switching off of the magnetic field down to the value $h=0$, the correlations between spins should change from the behavior typical of the universal regime to that of the nonuniversal regime, dominated by $C$ states. One may suspect, therefore, that the typical time interval of relaxation of physical properties, such as the pair correlation function of spins and the magnetization, would be radically different for $S=1/2$ and $S=1$. To be more specific, consider another physical property $\Gamma$, an effective interaction of two spin flips, which can be presented in the vicinity of the saturation field as a function of $S$ and $h_s-h$ [76]:

$$\Gamma(S,h)\sim\frac{2}{\sqrt{2}(\pi S\delta)^{-1}+1-1/S}, \quad \delta=\sqrt{4-\frac{h}{S}}. \tag{54}$$

As $h\to h_s$, $\Gamma$ remains positive for arbitrary spins and the system enters the universal regime, when all equilibrium physical properties for arbitrary spins could be derived from the ''Bose gas with repulsion'' model [76–78]. As $h$ is instantly set to zero, the magnetization should readjust itself from the values near to the saturation to the zero magnetization. The expression (54) for $\Gamma$ can still be used in this situation as an estimate of the effective interaction of two spin flips, at least at the initial stage of relaxation. Then, from Eq. (54) it follows that $\Gamma(S=1/2,h=0)\approx-3.64$, i.e., acquires the negative sign, which is different from $\Gamma(S=1,h=0)\approx 8.89$. The function $\Gamma(S,h=0)$ has a peak at $S=1$ and then saturates as $S\to\infty$ to the positive value 2. The negative sign of $\Gamma(S=1/2)$ means an attraction of two spin flips and rapid collapse of the magnetized state to a state without magnetization, while positive $\Gamma(S=1)$ means repulsion and much longer relaxation time for the magnetization. That is, we can expect an *anomalously slow relaxation* of the magnetization for spin 1, after an instant switching off of the magnetic field from the value close to saturation down to zero, compared to the case of spin 1/2.

### B. Ground-state energy

Spin-wave theory gives for the ground-state energy $E$ of the Heisenberg AF spin chains in the one-dimensional case the expansion in powers of inverse spin $1/S$ (see [79] and references therein),

$$E\simeq-S^2\left(1+\frac{\gamma}{2S}\right), \quad \gamma\approx 0.7. \tag{55}$$

The self-similarly renormalized expression, following from Eq. (55), is

$$E^*=-S^2\exp\left(\frac{\gamma}{2S}\tau\right) \tag{56}$$

and at $\tau=1$, $E^*(S=1)=-1.419$, approximating the ''exact'' numerical result $-1.401$ [80] with an accuracy of $1.285\%$, which is an improvement compared to the error $-3.64\%$ of the expression (55), corresponding to ''bare'' spin waves. For $S=2$, $E^*=-4.765$, in excellent agreement with the exact numerical result $-4.761$ [81].

The error, calculated for the renormalized expression (56) for $S=1/2$, is equal to 13.54%, becoming much worse than $-4.06\%$ for the bare spin waves, as compared to the exact value $E_0=-0.443\,15$ [30]. An attempt to improve the result for $S=1$, choosing the effective time $\tau$ from the exact result at $S=1/2$, gives the error of $-5\%$ as $S=1$, suggesting that between $S=1/2$ and $S=1$ some new physical mechanism comes into play, invalidating our attempt to match smoothly the ground-state energies for quantum spins based only on the renormalization of the spin-wave formula. On the other hand, a successful estimation of the ground-state energy for $S=1,2$, based on a $1/S$ expansion, suggests that a similar mechanism works for all $S\geqslant 1$ and quite an accurate estimate can be obtained from the formula (56).

Motivated by the existence of exact results for the autocorrelation function, we assume that the ground-state energy could be expanded around $E_0$ in powers of $S-\frac{1}{2}$, i.e., introduce a trial $S-\frac{1}{2}$ expansion around the exact solution at $S=\frac{1}{2}$:

$$E\sim-\left[|E_0|+A\left(S-\frac{1}{2}\right)\right],\quad S\to\frac{1}{2}.\tag{57}$$

The coefficient $A$ will be determined by matching Eq. (57) with the expression for the ground-state energy $E$, as $S\to\infty$:

$$E\sim-S^2\quad(S\to\infty).\tag{58}$$

Following the standard prescriptions of Sec. II, we obtain the left crossover approximation

$$E^*=-\left[\sqrt{|E_0|}+\left(S-\frac{1}{2}\right)\right]^2,\quad A=2|E_0|^{-1/2},\tag{59}$$

with $E^*(S=1)=-1.359$, approximating the exact result with the percentage error of $-3\%$, being only slightly better than the spin-wave result. In order to check the idea about similar mechanisms, forming the ground-state energy for $S\geqslant 1$, we rewrite Eq. (59) in the form

$$E^*=-[S^2+(2\sqrt{|E_0|}-1)S+(|E_0|-\sqrt{|E_0|}+1/4)]\tag{60}$$

and consider Eq. (60) as another form of the $1/S$ expansion. Applying to Eq. (60) the procedure of self-similar renormalization, we obtain

$$E^{**}=-S^2\exp\left(\frac{2\sqrt{|E_0|}-1}{S}\right),\tag{61}$$

with $E^{**}(S=1)=-1.393$ and the error $-0.57\%$. For $S=2$, $E^{**}=-4.72$ and the error is equal to $-0.86\%$. We again conclude that for the ground-state energy a simple crossover formula exists, covering the region from large spins to the small quantum spin $S=1$.

### C. Haldane gap

Haldane [5] conjectured the existence of radically different elementary excitation spectra for arbitrary integer and half–odd-integer one-dimensional Heisenberg spins, the former case being gapped with the smallest value of the gap $\Delta$ at $k=\pi$, while the latter case is gapless. In the limit of large $S$, Haldane used an approximate mapping onto the $O(3)$ nonlinear $\sigma$ model, leading to the following behavior of the gap [5]:

$$\Delta\sim S^2\exp(-\pi S),\quad S\to\infty.\tag{62}$$

Strictly speaking, the formula (62) describes only the ''slow'' part of the full dependence and does not take into account the ''fast'' part, describing the gap oscillations with changing spin, with zeros at half–odd-integer spins and maxima at integer values. Nowadays, it is established beyond a reasonable doubt [82] that for the half–odd-integer spins

$$\Delta\equiv 0\quad\left(S=\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2}\ldots\right).\tag{63}$$

For small integer spins $S=1,2$, the values of the gap are known from extensive numerical calculations. We suggest below a simple way to estimate $\Delta$ for arbitrary integer spins, based on the self-similar renormalization of a trial $S-\frac{1}{2}$ expansion for the Haldane gap and the knowledge of the asymptotic form (62), as $S\to\infty$, together with the demand for the absence of the gap for half–odd-integer spins.

Let us write the trial expansion for the gap in the vicinity of the point $S=\frac{1}{2}$ in the following form, satisfying the condition $\Delta(S=\frac{1}{2})=0$:

$$\Delta\sim a_2\left(S-\frac{1}{2}\right)^2+a_3\left(S-\frac{1}{2}\right)^3+a_4\left(S-\frac{1}{2}\right)^4+a_5\left(S-\frac{1}{2}\right)^5+\cdots\quad\left(S\to\frac{1}{2}\right),\tag{64}$$

where $a_k$ are positive. Following the general prescriptions of Sec. II, one can self-similarly renormalize Eq. (64) to the form

$$\Delta^*=a_2\left(S-\frac{1}{2}\right)^2\left[1-C_1\left(S-\frac{1}{2}\right)\right]^{n_1}\times\exp\left\{C_2\left(S-\frac{1}{2}\right)^2\left[1-C_3\left(S-\frac{1}{2}\right)\right]^{n_2}\right\}.\tag{65}$$

We require that Eq. (65) agree with Eq. (62), as $S\to\infty$, and also that at $S=3/2$, $\Delta=0$. Choosing the unknown coefficients and powers in Eq. (65) so as to satisfy the required conditions, we arrive at

$$\Delta^*=\left(S-\frac{1}{2}\right)^2\exp\left[-\frac{\left(S-\frac{1}{2}\right)^2}{S-\frac{3}{2}}\right],\tag{66}$$

where the value at $S=\frac{3}{2}$ is defined as the limit from the right, $S\to\frac{3}{2}+0$. At $S=1$, $\Delta^*=0.412$, in good agreement with the exact numerical value 0.4105 [80]; at $S=2$, $\Delta^*=0.025$, agreeing by an order of magnitude with the numerical value 0.085(5) [81] and in better agreement with the value 0.05, quoted in Ref. [83]. The formula (66) can be generalized by requiring that the exponential in Eq. (66) should have the form of an expansion in $1/S$ as $S\to\infty$ and zeros at $S=\frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots$, which yields

$$\Delta^*=\left(S-\frac{1}{2}\right)^2\exp\left[-\left(\frac{\left(S-\frac{1}{2}\right)^2}{\left(S-\frac{3}{2}\right)}+\frac{\left(S-\frac{3}{2}\right)^3}{\left(S-\frac{5}{2}\right)^3}+\frac{\left(S-\frac{5}{2}\right)^4}{\left(S-\frac{7}{2}\right)^5}+\frac{\left(S-\frac{7}{2}\right)^5}{\left(S-\frac{9}{2}\right)^7}+\ldots\right)\right], \tag{67}$$

where again the values at $S=(2n+1)/2$ are defined as the limits from the right. The value of the gap at $S=1$, given by Eq. (67), remains practically the same as above, while at $S=2$, $\Delta^*=0.068$. The gap, when described by the formula (67), practically vanishes for all integers $S\geqslant 3$, in agreement with the conclusion of Ref. [81].

### D. Other characteristics

Self-similar approximants can be constructed for other characteristics as well. Here we briefly mention only a couple of examples. Staggered magnetization $\Sigma$ of the antiferromagnetic anisotropic Ising-Heisenberg model of spin 1/2, as a function of the anisotropy parameter $\gamma$ (equal to zero for the Ising and one for the Heisenberg model), can be presented as expansion, valid at small $\gamma$ [84,85],

$$\Sigma(\gamma)\simeq 1-\gamma^2-\frac{1}{4}\gamma^4-\cdots. \tag{68}$$

At $\gamma=1$, according to Ref. [85], the long-range order parameter $\omega_\infty=\Sigma^2(\gamma)$ should disappear, i.e., $\Sigma(1)=0$. Let us continue the expression (68), from the region of $\gamma\ll 1$, to the region of $\gamma\sim 1$, satisfying the boundary condition for the disappearance of the long-range order. Then the left crossover approximation

$$\Sigma^*(\gamma)=(1-\gamma^2)\exp\left(-\frac{1}{4}\gamma^4\right) \tag{69}$$

satisfies the right boundary condition. Comparing $\omega_\infty^*=[\Sigma^*(\gamma)]^2$ with the extrapolation of numerical data, presented in Fig. 30 of Ref. [85], we found that they almost coincide.

At zero temperature, the dispersion, known from the linear spin-wave theory, is modified by the factor $Z$ in the spin-wave velocity and the expansion for $Z$ in powers of the inverse coordination number $1/z$ was obtained [86,87]:

$$Z\simeq 1+\frac{1}{4Sz}+\frac{3}{16Sz^2}+\cdots.$$

We continue this expression from the small values of $1/z$ to arbitrary $z$, while the value of spin is fixed, and determine the effective time $\tau$ from the exact value of $Z=\pi/2$, at $S=1/2$, $z=2$ [87]. The left crossover approximation has the form

$$Z^*=1+\frac{1}{4Sz}\exp\left(\frac{3}{4z}\tau\right), \quad \tau=2.21. \tag{70}$$

At $S=1/2$, $z=4$ (square lattice) we obtain from Eq. (70) $Z^*=1.189$, in excellent agreement with the results obtained by different methods [87]. At $S=1/2$, $z=6$, the case corresponding to a simple cubic lattice, $Z=1.11$, i.e., the quantum corrections to the spin velocity remain important.

## VI. COLLECTIVE EXCITATIONS

The knowledge of the elementary excitation spectrum is one of the key points for the description of many-body problems. Dealing with this extremely complicated problem, one often encounters the situation when the elementary excitation spectrum $\omega(k)$ is known for two different regions of the wave vector $k$. In the hydrodynamic region $k\to 0$, the form $\omega(k)$ could be determined either from experiment or theoretically. In the short-wavelength region $k\to\infty$, a dispersion corresponding to free particles should recover. Using the self-similar renormalization, it is possible to reconstruct $\omega(k)$ for arbitrary $k$. Consider some problems of this kind frequently occurring in condensed matter physics.

### A. Bogolubov spectrum

The case of a linear in $k$ spectrum, as $k\to 0$, and of quasifree massive particles, as $k\to\infty$, is of the most general type when Bose systems are considered. This kind of behavior is inherent to Bose systems and does not depend on the details of an interaction potential [6]. Consider the case of an anomalous sound dispersion, corresponding to an instability of the spectrum, as $k$ increases [7]. The following asymptotic expressions are available:

$$\omega(k)\simeq ck(1+\gamma k^2), \quad \gamma>0 \quad (k\to 0),$$

$$\omega(k)\simeq\frac{k^2}{2m^*} \quad (k\to\infty). \tag{71}$$

Here $c$ is the velocity of sound, $\gamma$ is responsible for the instability of the spectrum, and $m^*$ is the effective mass. The left crossover approximation can be derived following the standard prescriptions of Sec. II, which gives a result identical to the Bogolubov spectrum of a weakly nonideal Bose gas:

$$\omega^*(k)=ck\sqrt{1+\left(\frac{k}{2m^*c}\right)^2}. \tag{72}$$

Note that in distinction from the microscopic Bogolubov approach, valid for a diluted Bose system, the formula (72) may be used for arbitrary densities, assuming that the parameters $c$ and $m^*$ are taken from experiment. It looks rather intriguing that the same formula (72), which is usually derived with some lengthy calculations, can be immediately obtained by self-similarly interpolating the simple asymptotic expression (71).

### B. Liquid helium spectrum

Consider the case when more terms in the hydrodynamic limit are available, but the ''anomalous'' dispersion coefficient $\gamma$ is very small. Also, free particles are replaced by quasifree ''dressed'' particles with an effective mass $m^*$. The asymptotic behavior of the spectrum is

$$\omega(k) \simeq ck(1+\gamma k^2-\delta k^4), \quad \gamma\approx 0, \quad \delta>0 \quad (k\to 0),$$

$$\omega(k)=\frac{k^2}{2m^*} \quad (k\gg 1\,\text{Å}^{-1}). \tag{73}$$

This situation is typical for liquid $He^4$, where $\gamma=0\pm 0.05$ Å$^2$, $\delta=0.29\pm 0.03$ Å$^4$ [88], and $m^*=2-3m(He^4)$ [89]. The crossover approximant derived from Eq. (73) reads

$$\omega^*(k)=ck\left[[\exp(-\delta k^4)]^6+\left(\frac{k}{2m^*c}\right)^6\right]^{1/6}. \tag{74}$$

The expression (74) generalizes the Bogolubov spectrum (72). The main difference originates from the region of the intermediate $k\sim 1$. The formula (74) describes the experimental data for the elementary excitation spectrum of liquid $He^4$ [88] both qualitatively, predicting the existence of a roton minimum even for the bare mass $m^*=m(He^4)$, and quantitatively, with the maximal percentage error of about 20%. The value of the effective mass $m^*=2-3m(He^4)$ may have some relation to the formation of two-particle and three-particle correlated states [90]. We took above for estimates the value of the sound velocity equal to $2.4\times 10^4$ cm/s.

Our approach to deriving the spectrum for arbitrary $k$ corresponds to the Feynman approach [65], in which only the information about the short-wavelength and long-wavelength parts of the structure factor $S(k)$ are used. Then, instead of the phenomenological Feynman formula $\omega(k)=k^2/2mS(k)$, we apply the self-similar renormalization. The result is a Bogolubov-type formula. Thus a bridge between the Bogolubov and Feynman approaches to the spectrum of Bose systems [91] is established. The formula (74) is better qualitatively than the original Bogolubov spectrum (71), since it predicts the maxon-roton region, and better quantitatively than Feynman formulas, especially in the roton region. Here the Feynman formula works with an error of about 100%, while Eq. (74), in the worst case, gives an error about 10%. Our formula (74) is a three-parameter representation of the spectrum of liquid $He^4$, with parameters $c$, $\delta$, and $m^*$ coming from the regions of long-, intermediate-, and short-wavelengths, respectively.

The case of a stable soundlike spectrum, as $k\to 0$, and of quasifree particles, as $k\to\infty$, can also correspond to a collective-excitation branch in liquid $He^3$ [92]. The following asymptotic expressions are available:

$$\omega(k)\simeq ck(1-|\gamma|k^2) \quad (k\to 0),$$

$$\omega(k)\simeq\frac{k^2}{2m^*} \quad (k\to\infty). \tag{75}$$

In analogy to the case of $He^4$, we find the spectrum

$$\omega^*(k)=ck\left[[\exp(-|\gamma|k^2)]^4+\left(\frac{k}{2m^*c}\right)^4\right]^{1/4}. \tag{76}$$

However, since phonons in liquid $He^3$ are intertwined, in the region of intermediate wave vectors, with other collective excitations, it is impossible to observe rotons.

### C. Spectrum with a gap

Assume that the spectrum has a gap, as $k\to 0$, and possesses a minimum at this point, while, as $k\to\infty$, it becomes linear:

$$\omega(k)\simeq\Delta+\alpha k^2, \quad \alpha>0 \quad (k\to 0),$$

$$\omega(k)\simeq vk \quad (k\to\infty). \tag{77}$$

The left crossover approximation can be readily obtained, leading to the expression

$$\omega^*(k)=\Delta\sqrt{1+\left(\frac{vk}{\Delta}\right)^2}, \tag{78}$$

analogous to the spectrum of the Bardeen-Cooper-Schrieffer model of superconductivity.

### D. Dynamical scaling

The characteristic frequency $\omega_c(\zeta,k)$, appearing in the dynamical scaling hypothesis [8,93] and proportional to an inverse characteristic relaxation time of an order parameter, has two asymptotic forms, depending on the ratio $k/\zeta$, where $\zeta$ stands for the inverse correlation length. We shall discuss below only the behavior of density-density correlations in liquid systems [8]. Asymptotic expansions in the hydrodynamic regime $(k/\zeta\ll 1)$ and in the fluctuation regime $(k/\zeta\gg 1)$ are known:

$$\omega_c(\zeta,k)\simeq D_Tk^2\left[1+B\left(\frac{k}{\zeta}\right)^2+\cdots\right], \quad B>0 \quad \left(\frac{k}{\zeta}\ll 1\right),$$

$$\omega_c(\zeta,k)\simeq Ak^z\left[1+A'\left(\frac{\zeta}{k}\right)^2+\cdots\right] \quad \left(\frac{k}{\zeta}\gg 1\right), \tag{79}$$

where $D_T$ is the thermal diffusivity and $z$ is the dynamical critical index, which cannot be determined self-consistently within the framework of the dynamical scaling. The value of $B$ is estimated as $B=1$ [94] or $B=3/5$ [95].

Assume that the values of $z$ and $A$ are known. Then one can reconstruct the analytical expression for the characteristic frequency for arbitrary $k/\zeta$, obtaining the left crossover approximation

$$\omega_c^*(\zeta,k)=D_Tk^2\left[1+C\left(\frac{k}{\zeta}\right)^2\right]^n, \tag{80}$$

where

$$C=\zeta^2\left(\frac{A}{D_T}\right)^{2/(z-2)}, \quad n=\frac{z}{2}-1.$$

For $z=3$ [8] we obtain $n=1/2$. If now we plug into the expression (80) the dependences of $D_T \sim \epsilon^{\gamma-\alpha}$ and $\zeta \sim \epsilon^{\nu}$ on the distance $\epsilon$ from the critical point [8], then we recover immediately the well-known relation between the critical indices $z$, $\gamma$, $\nu$, and $\alpha$, that is, $z=2+(\gamma-\alpha)/\nu$ (all definitions are standard and may be found in Ref. [8]), which represents one of the central results of the dynamical scaling hypothesis. From this scaling relation, the dynamical critical index could be estimated from the values of the three other indices.

## VII. CONCLUSION

We suggested a general approach to describing crossover phenomena of arbitrary nature. The approach permits one to construct an accurate approximation for a function in the whole domain of its variable from asymptotic expansions near the boundaries. The minimal information needed to obtain a self-similar interpolation formula is two terms of an expansion near one of the boundaries and the limiting value at another boundary. Having only three such terms, it is already possible to get a reasonable approximation for the sought function in the total crossover region. When more terms are available, the procedure may be continued, improving the accuracy of approximations. An important feature of the method is that the self-similar crossover approximants always preserve the correct structure of the asymptotic expansions at both boundaries of the interpolation region. This is a clear advantage of the self-similar approach as compared to often used heuristic interpolations that may spoil the structure of the asymptotic expansions.

The possibility of obtaining accurate approximations from extremely scarce information, when no other methods work, is based on the following three points: (i) the idea of a self-similar renormalization group treating the transfer from one approximation to another as the evolution of a dynamical system, the approximation cascade; (ii) the requirement that this evolution be invariant with respect to algebraic transformations; and (iii) the use of control functions providing the stability and convergence of procedure.

Control functions introduced under the algebraic self-similar renormalization play, for the crossover problem, the role of effective crossover indices and effective crossover times. Depending on whether we start the renormalization procedure from an expansion either near the left or near the right boundary, we may distinguish the left and right crossover indices and, respectively, the left and right crossover times. Similarly, the resulting expressions for the sought function may be called the left and the right crossover approximations.

The form of the resulting self-similar approximations depends on the properties of the asymptotic expansions used. Mathematically equivalent expansions lead to the same form of crossover approximations. For example, compare the ground-state energy of the Fröhlich polaron as a function of the coupling parameter and the spectrum of collective excitations as a function of the wave vector. The weak-coupling series in powers of the coupling parameter is analogous to the long-wavelength spectrum in powers of the wave vector. The strong-coupling limit for the optic polaron is similar to the short-wavelength limit for the collective spectrum. As a result, the crossover approximation for the polaron energy has the same dependence on the coupling parameter as the crossover approximation for the collective spectrum on the wave vector. Thus, physically different quantities may have the same mathematical representation as a function of the corresponding variables. Keeping this in mind, we may say that there exist *classes of universality* of crossover phenomena.

It is worth emphasizing that the crossover approximations derived by applying the approach developed usually combine good accuracy with simplicity. This suggests that the self-similar renormalization provides a natural tool for extracting the maximal information from very short perturbative series that are impossible to analyze by other methods. Moreover, this makes us think that self-similarity, in some sense, is hidden in asymptotic series. This is why the self-similar renormalization becomes a natural effective tool of extracting such hidden information. The different physical examples presented in this paper prove as well that this is also a general tool applicable to arbitrary crossover phenomena.

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